Tautological Necessity and Tautological Validity With Quantifiers

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Some sentences containing quantifiers are *truth table necessary*.

That is, they are forced to be true just in virtue of the meanings of their connectives.

For instance,

$$\exists x\ \text{Tet}(x) \lor \neg\exists x\ \text{Tet}(x)$$

We don't have to know anything about the meanings of the FOL predicates (LeftOf, SameSize, etc.), names (a, b, etc.), or quantifiers or variables in order to tell that they have to be true.

What method can we use to determine whether a sentence is truth table necessary, even though it has quantifiers in it?

What we have to do is find a way of *feigning ignorance*.

We need to make sure that we ignore everything that Boole doesn't understand.

Once we ignore all of that stuff, we will have discovered the sentence's "Truth Functional" form.

We'll be seeing the sentence as Boole sees it.

We will call the method we use to uncover this form:

Boxing Up, Tagging, & Replacing

(The book has the same idea on page 263. They call it the "Truth Functional Form Algorithm." They also underline instead of box. Whatever. Either is fine.)

The Method

Here's how the method works.

Start at the beginning of the sentence you're investigating.

Apply the following recipe, moving your finger from left to right through your sentence and repeating until you reach the end of the sentence.

Apply the following recipe, moving your finger from left to right through your sentence and repeating until you reach the end of the sentence.

1. If your finger is on a connective or a parenthesis, leave it unboxed and skip over it. Apply this recipe to the rest of the sentence.
2. If your finger is on the start of an atomic sentence, box up that atomic sentence. Move your finger to the end of the box, apply this recipe to the rest of the sentence.
3. If your finger is on a quantifier: Box up the entire quantifier statement, starting at the quantifier and extending to the end of its scope. Move your finger to the end of the box, and continue applying this recipe to the rest of the sentence.

An Example: Boxing Exercise 10.1, sentence #4

From Exercise 10.1

4. ∀x (Cube(x) ∧ Small(x)) → ∀x (Small(x) ∧ Cube(x))

We start at the quantifier: ∀. The scope of that quantifier extends to the end of the first complete wff after it.

That means that it extends this far:

4. [∀x (Cube(x) ∧ Small(x))] → ∀x (Small(x) ∧ Cube(x))

Boxing Exercise 10.1, sentence #4

4. [∀x (Cube(x) ∧ Small(x))] → ∀x (Small(x) ∧ Cube(x))

Now we move our finger to the end of that box, and keep moving right.

What comes next is the →. That is a connective, so we leave it unboxed.

But after that we come to another ∀. So we start boxing again, to the end of the scope of that quantifier. In this case, that means to the end of the sentence.

4. [∀x (Cube(x) ∧ Small(x))] → [∀x (Small(x) ∧ Cube(x))]

Tagging 10.1.#4...

Now, we have to tag each unique boxed up sentence part with a unique capital letter sentence abbreviation (A, B, C, etc.)

- Make sure that you use the same capital letter for two boxes **if** they contain exactly the same string of symbols.
- Make sure that you use the same capital letter sentence for two boxes **only if** they contain the same string of symbols.

Tagging 10.1.#4...

4. $\boxed{\forall x\ (Cube(x) \land Small(x))} \rightarrow \boxed{\forall x\ (Small(x) \land Cube(x))}$

The two boxes don't have exactly the same string of symbols inside. They are similar, but not identical.

It doesn't matter that they mean the same thing. In order to be replaced with the same capital letters, the symbols themselves have to be identical, letter for letter.

In these sentences, the order of "Cube(x)" and "Small(x)" is switched. So they have to get different sentence letter abbreviations.

4. $\boxed{\forall x\ (Cube(x) \land Small(x))}_A \rightarrow \boxed{\forall x\ (Small(x) \land Cube(x))}_B$

Note: When exercise 10.1 asks you to write an "Annotated Sentence" in your table, this is what it is looking for: The boxed up sentence with an appropriate assignment of capital letters.

...& Replacing (Exercise 10.4#4)

Now, to finish this off and get the truth functional form, just replace each box with its capital letter.

So our example of sentence #4 from Exercise 10.1 before ...

4. $\boxed{\forall x\ (Cube(x) \land Small(x))}_A \rightarrow \boxed{\forall x\ (Small(x) \land Cube(x))}_B$

...becomes ...

4. $A \rightarrow B$

That is sentence 4's **truth functional form**. That is what you should write in the second column of the table you have to write up for Exercise 10.1, under sentence 4.

Is 10.1#4 a Truth Table Necessity?

Is a sentence that has this form a tautology—a Truth Table Necessity? That is, is it true on every row of its truth table?

4. $A \rightarrow B$

No. Clearly not. It will have a row on its truth table where A is true and B is false. On that row, $A \rightarrow B$ will be false. So it isn't a tautology.

That means that we don't want to write "a" in the third column of the table you are writing for Exercise 10.1.

Is 10.1.4 *Logically* Necessary?

Next question:

4. ∀x (Cube(x) ∧ Small(x)) → ∀x (Small(x) ∧ Cube(x))

- Is this sentence *logically* necessary?

Is 10.1.4 Logically Necessary?

4. ∀x (Cube(x) ∧ Small(x)) → ∀x (Small(x) ∧ Cube(x))

Answer: **Yes**

Why?

Because: If the antecedent (∀x (Cube(x) ∧ Small(x))) is true, that means that everything is a cube and small.

So the consequent (∀x (Small(x) ∧ Cube(x))) has to be true, too. Everything is a small cube.

That means that we will want to write "b" in the last column of the chart we're writing for Exercise 10.1.

Another Example: 10.1#1

1. ∀x (x=x)

How do we box it up?

Boxing Up 10.1#1

1. ∀x (x=x)

Start at the quantifier ∀. Its scope goes to the end of the first complete wff after the quantifier.

Here, that takes us all the way to the end of the whole sentence.

So here's what we get:

1. $\boxed{\forall x\ (x=x)}$

Now assign a capital letter sentence abbreviation.

1. $\boxed{\forall x\ (x=x)}_A$

(This is what we'll want to write in the first column of our chart for Exercise 10.1.1, under "Annotated sentence")

10.1.1 Truth Functional form

1. $\boxed{\forall x\ (x=x)}_A$

That means that the truth functional form of the sentence is just

1. A

(That's what we'll want to write in the second column of the chart we're writing to hand in for Exercise 10.1, under "Truth-functional form" for sentence number 1.)

10.1.1 Truth Table Necessary?

Question: So is sentence 10.1.1 a tautology?

To answer, we look at its truth table form again:

1. A

Answer: Heck, no. It's just a single simple sentence, according to Boole. So it will be true on some rows of the truth table, and false on others.

10.1.1 Logically necessary???

But is 10.1.1 a *logical* necessity?

That is, is it a sentence that is forced to be true when we pay attention to more than just the meanings of the connectives?

Once we start paying attention to the meanings of the connectives **and** the quantifiers, identity, and then the rest of the predicates, then the sentence goes back to looking like this:

1. $\forall x\ (x=x)$

Question: Does that sentence have to be true?

10.1.1 Logically Necessary

Well, what does it say?

1. $\forall x\ (x=x)$

- "Every object is one and the same object as itself."

There is just no way that could be false. Everything must be identical to itself.

So 10.1.1 is logically necessary.

Once again, we'll have to write "b" in the last column of our chart we're submitting for Exercise 10.1.

TT-Validity of Arguments

The Boxing, Tagging & Replacing technique will also help us to determine whether an argument involving quantifiers is truth table (or "tautologically") valid.

This is what you would be asked to do if you were assigned Exercise 10.4.

Extending the BTR method to Arguments

The only new wrinkle concerns the way we tag boxes with capital letters:

- When we are tagging our boxes with capital letter sentence names, we have to make sure that we use unique capital letters for different boxes across all the sentences in the argument.
- Also, if we have the same box reappearing in different places in the argument, we need to make sure that we use the same capital letter for that box wherever it appears in the argument.

Exercise 10.4

Let's Box, Tag, and Replace the argument in Exercise 10.4

$$\begin{array}{|l} \forall x\ \text{Cube}(x) \rightarrow \exists y\ \text{Small}(y) \\ \neg\exists y\ \text{Small}(y) \\ \hline \neg\forall x\ \text{Cube}(x) \end{array}$$

We just box this up the same way we boxed up sentences before.

How does that go?

The Argument in Exercise 10.4

$$\begin{array}{|l} \boxed{\forall x\ \text{Cube}(x)} \rightarrow \exists y\ \text{Small}(y) \\ \neg\exists y\ \text{Small}(y) \\ \hline \neg\forall x\ \text{Cube}(x) \end{array}$$

Boxing Up the Argument in Exercise 10.4:

$$\begin{array}{|l} \boxed{\forall x\ Cube(x)} \rightarrow \boxed{\exists y\ Small(y)} \\ \neg \boxed{\exists y\ Small(y)} \\ \hline \neg \boxed{\forall x\ Cube(x)} \end{array}$$

Notice that we skip over the beginning negations for the second premise and the conclusion. Boole can understand those, since they're not inside of any quantifiers.

Tagging Exercise 10.4

Now we have to assign capital letter sentence abbreviations.

Make sure that we use different letters for different boxes and the same letters for the repeating boxes!

See any repeats we need to watch out for?

$$\begin{array}{|l} \boxed{\forall x\ Cube(x)} \rightarrow \boxed{\exists y\ Small(y)} \\ \neg \boxed{\exists y\ Small(y)} \\ \hline \neg \boxed{\forall x\ Cube(x)} \end{array}$$

Exercise 10.4:

$$\begin{array}{|l} \boxed{\forall x\ Cube(x)}_A \rightarrow \boxed{\exists y\ Small(y)}_B \\ \neg \boxed{\exists y\ Small(y)}_B \\ \hline \neg \boxed{\forall x\ Cube(x)}_A \end{array}$$

Replace tagged boxes with their capital letter abbreviations ...

$$\begin{array}{|l} A \rightarrow B \\ \neg B \\ \hline \neg A \end{array}$$

...to reveal the truth-functional form of the argument.

Truth Functional Form and TT-Validity

This is what we would want to write out on paper if 10.4 were assigned as homework:

Exercise 10.4:

$$\begin{array}{|l} A \rightarrow B \\ \neg B \\ \hline \neg A \end{array}$$

And then we have to answer the question whether this argument is TT-valid.

What do you think?

Assessing TT-Validity

It's TT-Valid if and only if it doesn't have any Truth Table counterexamples.

A counterexample would be a row where the premises $A \rightarrow B$ and $\neg B$ are true, but the conclusion $\neg A$ is false.

Would there be any rows like that?

Nope. There would not be any rows like that.

Wherever $A \rightarrow B$ and $\neg B$ are true on the truth table, the conclusion $\neg A$ has to be true, too.

So there are no counterexample rows.

So the argument from Exercise 10.4 is TT-valid.

So we should write down (a) for our assessment of the argument: It is tautologically valid.

Assessing Logical Validity

Thus, it has to be Logically Valid, too.

(All TT-valid arguments are logically valid.)

Using Other Programs to Check

Still not sure whether this is TT-valid?

$A \rightarrow B$
$\neg B$

$\neg A$

You could check it by constructing a truth table in Boole.

The book tells you that you can use the rule of **Taut Con** to check your answers.

Taut Con

Taut Con is a wildcard rule in Fitch. It is kind of like **Ana Con**, only less powerful. It tells you whether a step is a TT-Consequence of other steps.

Ana Con tells us whether an argument is Logically Valid. It pays attention to the connectives, the quantifiers, identity, and meanings of all of the other predicates, too.

Taut Con, like Boole, only pays attention to the meanings of the connectives.

One more Exercise ...

Take a look at Exercise 10.2

It is not TT-valid. (Why?)

But it is logically valid. (Why?)

So if this Exercise were assigned, we would write (b) for our assessment.